A Qualitative Study of Dynamical Systems through One and Two-Dimensional Models

Dissertation submitted to

Visvesvaraya National Institute of Technology, Nagpur

in partial fulfillment of the requirements for the award of the degree

Master of Science

in

Mathematics

by

Jigyasa

(MS23MTH010)

Under the supervision of

Dr. Purnima Satapathy

Department of Mathematics

Visvesvaraya National Institute of Technology

Nagpur – 440010, India

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Declaration

I, Jigyasa, hereby declare that this dissertation titled "A qualitative study of dynamical systems through one and two-dimensional models" is carried out by me at the Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur. The work is original and has not been submitted earlier, in whole or in part, for the award of any degree/diploma at this or any other institution/university.

Jigyasa

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This is to certify that the dissertation entitled "A qualitative study of dynamical systems through one and two-dimensional models" is a bonafide work done by **Ms. Jigyasa** (Enrollment No. MS23MTH010). The project work has been completed under my guidance and supervision in a satisfactory manner. This work is a part of the requirement for the award of **Master of Science in Mathematics** from the Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur. This dissertation meets the standards and requirements for the award of the Master of Science degree.

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Acknowledgments

It is with deep sense of gratitude that I express my sincere thanks to my supervisor, Dr. Purnima Satapathy, Assistant Professor, Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur, for her invaluable guidance, encouragement, and constant support throughout my project work. Without her active mentorship, this work would not have been possible. Her profound academic insight, along with her generous support both academically and personally, are deeply appreciated.

I would also like to extend my heartfelt thanks to Dr. Pallavi Mahale, Professor and Head of the Department of Mathematics, Dr. G. Naga Raju, Associate Professor, and Dr. Jyoti Singh, Assistant Professor, Department of Mathematics, VNIT Nagpur, for providing excellent facilities and for their continued encouragement during the course of this project. I am grateful to the Project Assessment and Progress Committee members, especially Dr. M. Devakar, Professor, Department of Mathematics, VNIT Nagpur, for their valuable suggestions and support.

I sincerely appreciate the assistance and advice of Mr. Ashutosh Kumar Karna, Ph.D. research scholar, and Mr. Tapan Kumar Muduli, Ph.D. research scholar, Department of Mathematics, VNIT Nagpur, for their time, helpful discussions, and valuable recommendations.

I am forever indebted to my parents and family members for their unconditional love, blessings, and constant encouragement.Finally, I bow in reverence to the Almighty, whose blessings have enabled me to successfully complete this work.

Jigyasa

Abstract

This thesis provides a comprehensive review of the fundamental principles of dynamical systems, illustrated through the analysis of one and two-dimensional models. Beginning with a historical overview and essential preliminaries, the study introduces flows on a line, emphasizing fixed points, stability analysis, and the influence of parameters, culminating in the study of bifurcations. Real-world applications—such as population dynamics, mechanical oscillators, potential wells, and laser systems—are incorporated to demonstrate how various systems evolve over time. The investigation is then extended to two-dimensional systems, where linear systems, eigenvalues, and phase portraits are analyzed. The primary objective is to develop a clear and intuitive understanding of how dynamical systems behave, especially in scenarios where analytical solutions may not be attainable.

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Chapter 1

Introduction

In today's mathematical world, there is a great fascination for chaos and fractal as they represent hands-on mathematics that is alive and changing.The chaos and fractals are part of grandeur topic Dynamics. Dynamics is the study of change in the systems that evolve with time. We are exposed to dynamical systems such as chemical kinetics, differential equation, and so on.

1.1 History of Dynamics

Dynamics originally started as a part of physics in the seventeenth century. Isaac Newton played a vital role in the beginning of dynamical system. He introduced the concept of differential equation, explained the laws of motion and universal gravitation, and used them to describe the movement of planets. Newton solved two body problem, predicting the movement of earth around the sun because of gravity. When applying the same methods around three body problem, where earth, sun, and moon interact then after many trials, scientist realized that it was impossible to find simple formula for motion of three bodies.

A big change came in late 1800s with Poincaré's participation. He worked on the broader concept that instead of finding exact solution, one can see whether the solar system stay stable forever or if the planets could eventually fly away into space. He developed a new approach to study dynamics ,using geometry to know how systems evolve over time. Poincare

also discovered that some systems, even if they follow clear rules, can behave in unpredictable way which is now know as Chaos.

In early 1900s,scientist studied non linear oscillators, the system that repeat but not in simple ways. These studies played vital role in development of radars, lasers, and radios. Many researchers came up with new mathematical approach to understand these complex systems. At the same time, scientist such as Birkhoff, Kolmogorov, Arnold, and Moser expanded Poincaré's geometric idea to improve understanding of classical mechanics.

The invention of high-speed computers in the 1950s changed everything. From now, scientists could use computers to explore and demonstrate the behavior of complex systems. In 1963, Edward Lorenz, found chaotic behavior while studying a simple model of atmospheric convection. He noticed that tiny differences in starting conditions led to very different outcomes. This helped to explain why weather forecasts are often unreliable. When he plotted the behavior of the system, it formed a never ending butterfly-shaped pattern as shown in Figure 1.1, now called a fractal structure.

During the 1970s, chaos theory became very popular. Important advances included Ruelle and Takens' theory about turbulence, May's discovery of chaos in simple biological models. Feigenbaum's work connected chaos theory to ideas from physics about phase transitions. Researchers like Gollub, Libchaber, Swinney, Linsay, Moon, and Westervelt performed many experiments which confirmed chaotic behavior in fluids, chemical reactions, mechanical systems, and electronics. Besides the growth of chaos theory, Benoît Mandelbrot introduced the idea of fractals and showed their importance in science with striking computer images. In biology, Winfree used dynamic systems methods to study natural rhythms like the 24-hour circadian cycle and the heartbeat. By the 1980s, dynamics had grown into a large and active field, with research spreading into many areas of science and engineering. Now, dynamics is an important part in every natural phenomenon.

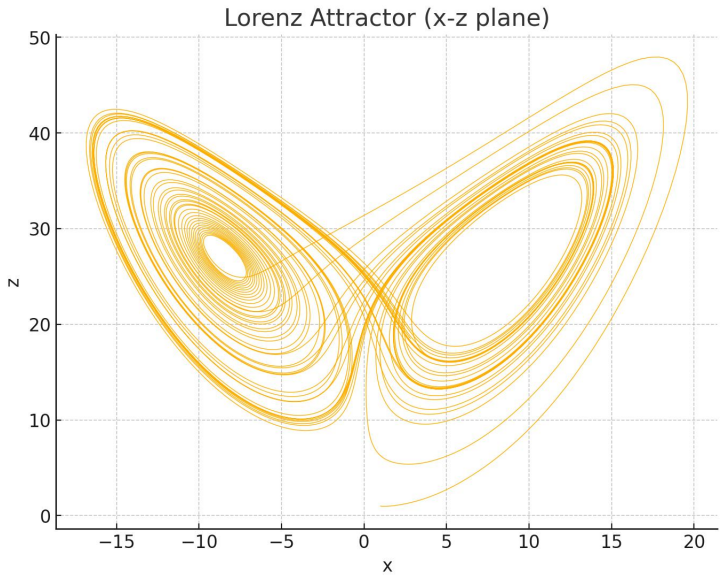


Figure 1.1: Fractal Structure

1.2 Non-linearity in dynamical systems

Dynamics works with two types of systems: differential equations and iterated maps. Differential equations describe how systems change continuously over time, while iterated maps deal with systems where time is discrete.

Among differential equations, we distinguish between ordinary and partial types. This thesis is confined on ordinary differential equations. The general system of ODEs is as follows:

$$\begin{aligned} \dot{x_1} &= f_1(x_1, x_2.....x_n), \\ \dot{x_2} &= f_2(x_1, x_2.....x_n), \\ &\vdots \\ \dot{x_n} &= f_n(x_1, x_2.....x_n), \end{aligned}$$

where each x_i represents a system variable, and the functions f_i define the system's behavior over time.

Linear systems have variables appearing only to the first power so they can be easily studied by breaking into parts and following the principle of superposition whereas, nonlinear systems

involve terms like products, powers, or trigonometric functions of the variables due to this a complex non linear system is very difficult to be solved analytically. Our everyday life exhibit non-linear behavior such as operation of laser, and turbulence of the fluid.

To study such complex systems, we use phase space - a multidimensional space used to represent all possible states of a dynamical system, with each solution tracing a trajectory in the space. Even for systems with explicit time-dependence, we can convert them into autonomous systems by adding extra dimensions, which helps in visualizing their behavior.

Chapter 2

One-Dimensional Flows

The general form of one dimensional system's looks like $\dot{x} = f(x)$, where $x(t)$ is the function of t and $f(x)$ is a smooth function of x.

2.1 Understanding Through Geometry

Consider the non linear differential equation:

$$\dot{x} = \sin x$$

We can find the exact solution of the equation. Solving analytically, we get

$$t = -\ln|\csc x + \cot x| + C$$

for evaluating the constant C, suppose that $x = x_0$ at $t = 0$. Hence the solution is

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|$$

The result is exact but how can we find the answers of the following:

- Suppose $x_0 = (\frac{\pi}{4})$, what happens to the solution when $t \to \infty$?
- For any arbitrary initial condition x_0, how the function $x(t)$ will behave as $t \to \infty$?

The analytical approach cannot work on this. In the graphical analysis of $\dot{x} = \sin x$, we plot the phase points with respect to state variables x and $\dot{x}$ (Figure 2.1). The flow of vector

field move towards the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$. This approach allow

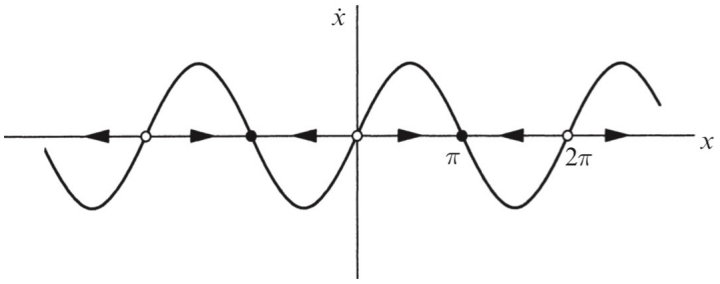


Figure 2.1: Phase space of $\dot{x} = \sin x$

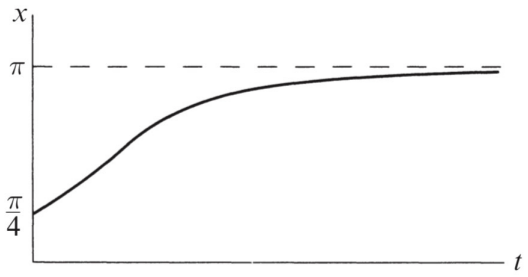


Figure 2.2: Variation of $\sin x$

us to answer the above questions asked: When the particle starting at $x_0 = (\frac{\pi}{4})$ moves to right faster until it reaches the maximum value of $\sin x$ i.e., $x_0 = (\frac{\pi}{2})$ then slows down and approaches $x = \pi$ as shown in figure 2.2. The same reasoning apply to second question.

2.2 Fixed Points And Stability of the system

In dynamical system fixed point is a point in phase space where the system remains at rest if it starts there. Mathematically when $x = x^*$ such that $f(x^*) = 0$ then x^* is considered a fixed point.

If a small change near the fixed point eventually bring the function back to fixed point then its stable fixed point. If a small change near the fixed point eventually decay the function to infinity then its unstable fixed point.

Example 2.1 *Consider the electrical circuit shown in figure* 2.3. *Suppose that there is no charge on the capacitor initially at* $t = 0$. *Sketch the graph of Q(t)and comment on stability.*

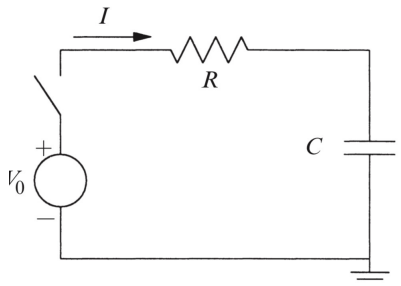


Figure 2.3: Electrical circuit

Solution: The total voltage drop must be zero, $-V_0 + IR + \dfrac{Q}{C} = 0$.
The current I causes charge to accumulate at a rate $\dot{Q} = I$. Then,

$$-V_0 + R\dot{Q} + \frac{Q}{C} = 0,$$
$$\dot{Q} = f(Q),$$
$$f(Q) = \frac{V_0}{R} - \frac{Q}{RC}$$

According to the figure 2.4 fixed points where $f(Q) = 0$ occurs at $Q^* = CV_0$ which is globally stable.

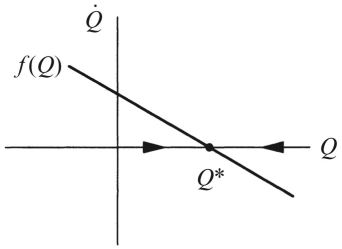


Figure 2.4: Phase space of charge and current

2.3 Population Growth

The simple model growth of population of organisms is $\dot{N} = rN$, where $N(t)$ is population at time t and $r \geq 0$ is growth rate. While exponential growth, given by $N(t) = N_0 e^{rt}$,

models early population increase, it does not account for limitations such as finite resources. In practice, as the population N increases, the per capita growth rate decreases due to overcrowding and limited availability of resources. To incorporate these effects, the logistic model is used:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right),$$

where r is the growth rate and K is the carrying capacity of the environment.

The equation predicts that the growth slows down as N approaches K, and becomes negative if $N > K$. This system has two fixed points shown in figure 2.5:

- $N = 0$ (unstable equilibrium),
- $N = K$ (stable equilibrium).

A phase line analysis //(Figure 2.6) shows that for any initial condition $N_0 > 0$, the population tends to K as time increases. However, if $N_0 = 0$, the population remains at zero, as the model has no generation spontaneously.

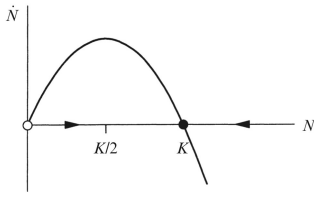


Figure 2.5: Phase space of population model

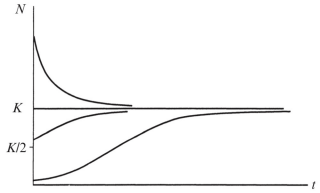


Figure 2.6: Variation of N with t

2.4 Analysis of linear stability

So far we found the qualitative behavior of system, often we want to move a more quantitative measure of stability, such as rate of decay to stable point.
Let x^* be a fixed point of the system $\dot{x} = f(x)$. To study the stability of this fixed point, consider a small perturbation $n(t) = x(t) - x^*$, representing a slight deviation from the equilibrium point.

Differentiating both sides with respect to time gives:

$$\dot{n} = \dot{x} = f(x^* + n).$$

Using Taylor's series expansion around x^*, we can approximate:

$$f(x^* + n) = f(x^*) + nf'(x^*) + \mathcal{O}(n^2).$$

Since $f(x^*) = 0$, this simplifies to:

$$\dot{n} = nf'(x^*) + \mathcal{O}(n^2).$$

Neglecting higher-order terms for small n, we obtain the linearized equation:

$$\dot{n} \approx nf'(x^*).$$

This equation describes the local behavior of the system near the fixed point and is known as linearization about x^*. The sign of the derivative $f'(x^*)$ determines the stability:

- If $f'(x^*) < 0$, then $n(t)$ decays exponentially with time. This implies that trajectories near x^* return to it, so the fixed point is stable.
- If $f'(x^*) > 0$, then $n(t)$ grows exponentially. Nearby trajectories move away from the fixed point, indicating it is unstable.
- If $f'(x^*) = 0$, the linear term vanishes and the higher-order terms $\mathcal{O}(n^2)$ become significant. In such cases, linear stability analysis is inconclusive, and one must consider the nonlinear behavior to determine stability.

The reciprocal $\frac{1}{|f'(x^*)|}$ is a characteristic time scale; it determines the time required for $x(t)$ to vary significantly in the neighborhood of x^*.

2.5 Impossibility of Oscillations in First-Order Systems

In first-order dynamical systems $\dot{x} = f(x)$, the dynamics are entirely governed by the nature of fixed points. Trajectories either approach a fixed point or diverge to infinity; oscillations and periodic solutions are not possible. This condition arises due to motion, as it is always monotonic—either strictly increasing, strictly decreasing, or constant—because the system evolves along a one-dimensional line without the possibility of turning back. Therefore, once the direction of motion is set by the sign of $f(x)$, it cannot reverse, and hence closed orbits cannot form.

Mechanical Analogy of Overdamped Systems

To further understand the absence of oscillations in first-order systems. Imagine a mass m attached to a nonlinear spring and immersed in a highly viscous fluid, such as oil or honey. The restoring force exerted by the spring is given by $F(x)$, and the damping force from the fluid is proportional to velocity, $b\dot{x}$. According to Newton's law, the motion of the mass is described by:

$$m\ddot{x} + b\dot{x} = F(x).$$

In the limit of strong damping, where $b\dot{x} \gg m\ddot{x}$, the inertia term becomes negligible. The equation simplifies to:

$$b\dot{x} \approx F(x) \quad \Rightarrow \quad \dot{x} = \frac{1}{b}F(x) = f(x),$$

which matches the form of a first-order system. In this overdamped regime, the mass slowly relaxes to equilibrium without overshooting. The damping prevents any oscillatory motion, and the system returns to equilibrium monotonically.

2.6 Potentials

An alternative way to understand the dynamics of a first-order system $\dot{x} = f(x)$ is through the concept of a potential function. We define a potential $V(x)$ such that:

$$f(x) = -\frac{dV}{dx}.$$

This means the system behaves like a particle moving in a potential landscape, where the particle always slides downhill due to strong damping. Because the motion is overdamped, inertia is negligible, and the particle simply follows the slope of the potential.

We compute the time derivative of the potential:

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt} = \frac{dV}{dx} \cdot \dot{x} = -\left(\frac{dV}{dx}\right)^2 \leq 0.$$

This shows that $V(x(t))$ always decreases with time unless the particle is at a fixed point (where $\frac{dV}{dx} = 0$). Hence, the system moves toward lower potential values.

In this interpretation:

- Stable fixed points correspond to local minima of $V(x)$, where the particle settles down to equilibrium.
- Unstable fixed points correspond to local maxima, where any small disturbance will push the particle away.

2.7 Bifurcation in 1-D flow

We have seen so far that in one-dimensional flow the system either settles to equilibrium or goes away to infinity, so what makes interesting to study 1-D flow is dependence on parameters. The parameters are the external values that influence the behavior of the system but do not change their value over time. Bifurcation is the qualitative changes i.e., stability may changes, that occurs in system due to variation of parameter and the parameter values at which it occurs are bifurcation points.

A simple example of bifurcation is the buckling of a vertical beam. When a small weight is

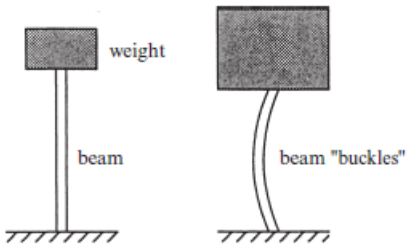


Figure 2.7: Buckling of a vertical beam under increasing weight.

placed on top, the beam stays straight and vertical—this is a stable state. But as the weight increases beyond a certain point, the beam can no longer stay upright and bends to one side (Figure 2.7).

Here:

- The weight is the *control parameter*.
- The sideways deflection of the beam is the *dynamical variable* x.

2.7.1 Saddle Node Bifurcation

The saddle node bifurcation is the basic mechanism by which fixed points are created and destroyed.
Example:

$$\dot{x} = r + x^2$$

where r is a real parameter.

As the parameter r increases toward zero from below, two fixed points move closer. At $r = 0$, they merge into a single, half-stable fixed point at $x^* = 0$. For $r > 0$, this fixed point disappears completely.

This sudden change in the number and stability of fixed points is called a **bifurcation**. The system's behavior is qualitatively different for $r < 0$, $r = 0$, and $r > 0$, making $r = 0$ the bifurcation point. In the Figure 2.8, we can clearly observe that how varying r with time can create as well as destroy the fixed points.

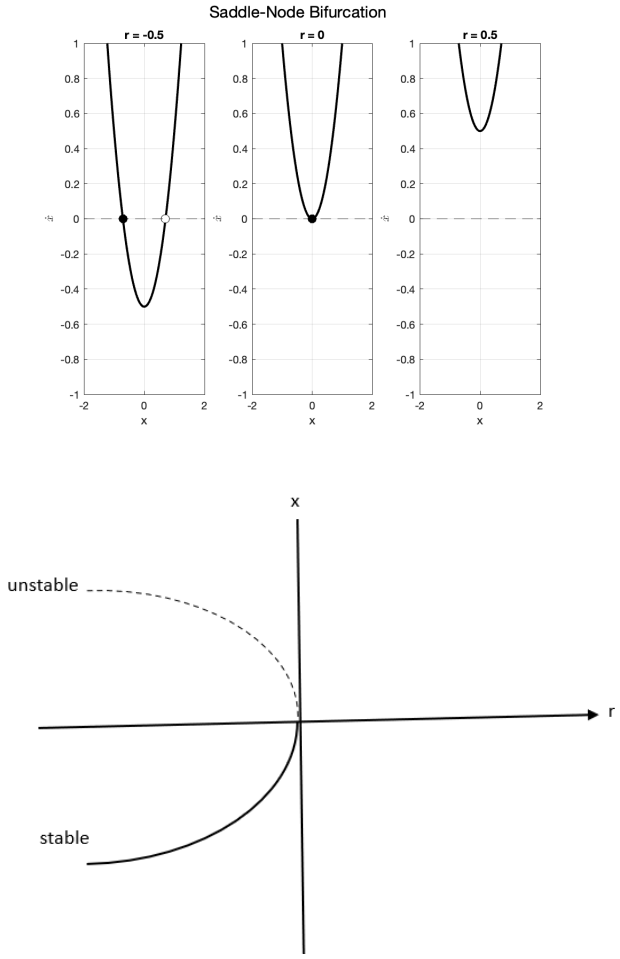


Figure 2.8: Phase Space with varying paramter r

Normal form: Near to all saddle-node bifurcation, the dynamics look the same as $\dot{x} = r+x^2$ or $\dot{x} = r - x^2$ and it is known as Normal form of saddle node bifurcation.

Algebraic approach to saddle-node bifurcation: $\dot{x} = f(x, r)$ near the bifurcation at $x = x^*$ and $r = r_c$

Taylor expansion,

$\dot{x} = f(x^*, r_c)+(x-x^*)\left[\frac{\partial f}{\partial x}\right]_{(x^*,r_c)}+(r-r_c)\left[\frac{\partial f}{\partial r}\right]_{(x^*,r_c)}+\frac{1}{2}(x-x^*)^2\left[\frac{\partial^2 f}{\partial x^2}\right]_{(x\cdot r_c)}+O(r^2)+O(x^3).....$

Key conditions: $f(x^*, r_c) = 0$ and $\left[\frac{\partial f}{\partial x}\right]_{(x^*,r_c)} = 0$

Reduced form: $\dot{x} = a(r - r_c) + b(x - x^*)^2$

Example 2.2 *Show that the first order system $\dot{x} = r - x - e^{-x}$ undergoes saddle node bifurcations as r is varied.*

Solution: So the given equation is $f(x) = r - x - e^{-x}$, for bifurcation points we put $f(x) = 0$.

$$e^{-x} = r - x,$$
$$\frac{d}{dx}e^{-x} = \frac{d}{dx}(r - x),$$

bifurcation point is $r^* = 1$ at $x = 0$.

In the figure 2.9, we see that varying r caused the deflection in fixed points.

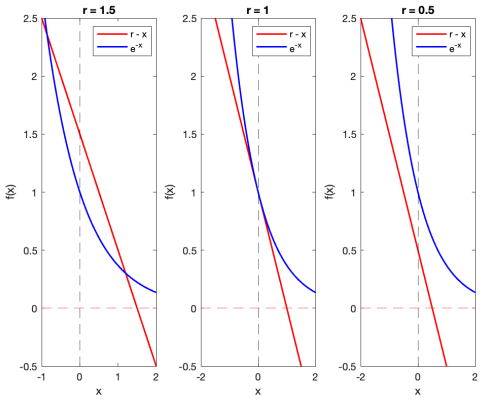


Figure 2.9: Phase space of saddle node bifurcation with varying parameter r

2.7.2 Transcritical Bifurcation

It is a type of bifurcation in which two fixed points intersect and exchange stability but never be destroyed as a parameter r is varied.

Normal form: $\dot{x} = rx - x^2$, the two fixed points are $x = 0$ and $x = r$

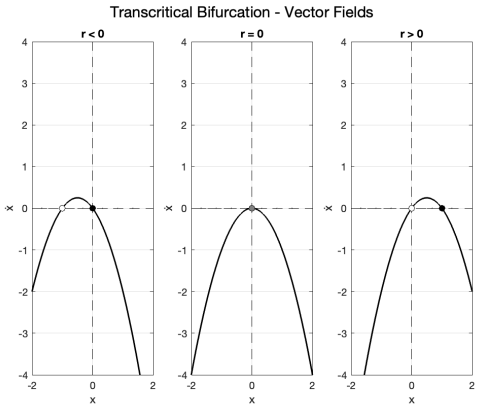


Figure 2.10: Phase space with parameter r

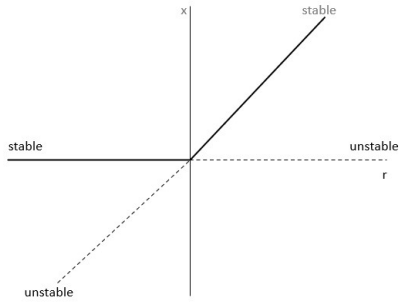


Figure 2.11: Phase space of transcritical bifurcation with parameter r

Example 2.3 *Show that* $\dot{x} = x(1-x^2) - a(1-e^{-bx})$ *system undergoes a transcritical bifurcation at* $x = 0$.

Solution:

- x = 0 is always a fixed point for any values of a and b.
- Near $x = 0$, expand $1 - e^{-bx}$ as:

  $1 - e^{-bx} \approx bx - \frac{1}{2}b^2x^2$

  .

  Substituting gives:

  $\dot{x} \approx (1-ab)x + \frac{1}{2}ab^2x^2$.

- Bifurcation Condition: $ab = 1$, the bifurcation curve.
- Approximate Nonzero Fixed Point:

  $x^* \approx \frac{2(ab-1)}{ab^2}$.

2.7.3 Threshold of Laser

We now apply our mathematical framework to a simplified physical system—a laser—based on the treatment by Haken (1983).

Specifically, we consider a solid-state laser, where laser-active atoms are embedded in a solid matrix enclosed by partially reflective mirrors. An external pump supplies energy to excite these atoms from their ground states.

When the pump is pushed a little the atoms get excited and travel in different directions emitting lamp light whereas after a certain amount of external force known as threshold, atoms start travelling in same phase like beam and start acting as a laser.

There is also the loss of atoms for the boundary of the solid state matrix.

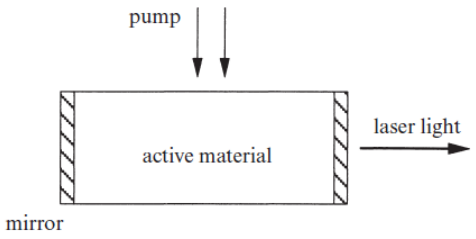


Figure 2.12: Solid-state laser

- The dynamical variable is the number of photons $n(t)$.
- Its rate of change is $\dot{n} = gain - loss = GnN - kn$.
- Then $\dot{n} = Gn(N_0 - \alpha n) - kn = (GN_0 - k)n - (\alpha G)n^2$ undergoes a trans-critical Bifurcation when $N_0 = k/G, n^* = 0$.

2.7.4 Pitchfork Bifurcation

In this type of bifurcation, the fixed points tends to appear and disappear in symmetrical pairs.

Supercritical pitchfork Bifurcation

A stable fixed point becomes unstable and two new stable fixed points emerge symmetrically on varying the parameter.
Example:

$$\dot{x} = rx - x^3$$

where r is parameter.

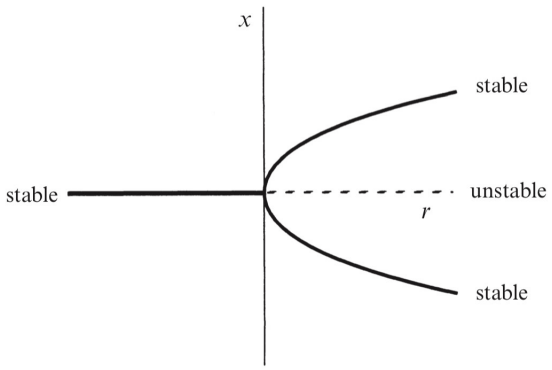


Figure 2.13: Phase space of supercritical pitchfork birfurcation with varying r

Example 2.4 *Show that* $\dot{x} = -x + \beta \tanh x$ *undergoes supercritical pitchfork bifurcations as* β *is varied.*

Solution: $f(x) = -x + \beta \tanh x = 0$.

- $\beta < 1$,*the only fixed point* $x = 0$
- $\beta = 1$,*the only fixed point* $x = 0$
- $\beta > 1$,*the fixed point are* $x = 0$ *and two stable symmetrical fixed points.*

bifurcation point is $\beta^* = 1$ *at* $x = 0$.

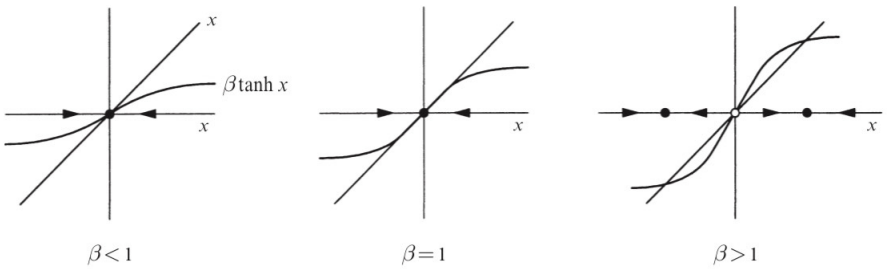


Figure 2.14: Phase Space when β is varied

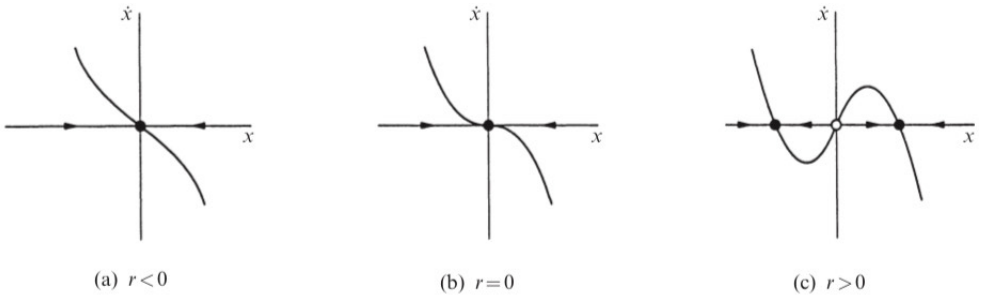
Subcritical pitchfork Bifurcation

Stable equilibrium point becomes unstable as parameter is varied and a new pair of unstable points emerges in a discontinuous manner.

Example:

$$\dot{x} = rx + x^3,$$

where r is parameter.



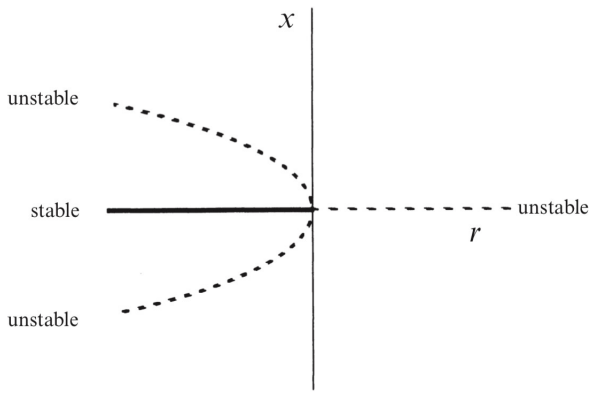


Figure 2.15: Phase space of subcritical pitchfork bifurcation with parameter r

Example 2.5 $\dot{x} = rx + x^3 - x^5$

Solution: Visualing the figure 2.16, we can draw the following conclusions:

- For small r, origin is stable and two unstable branches emerge for $r < 0$.
- As small r decreases, $r_s < r < 0$, unstable branches become stable.
- When $r = r_s$, saddle node bifurcation occurs.

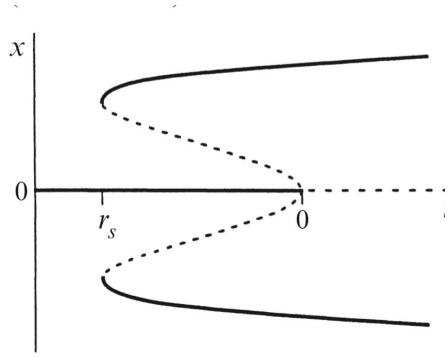


Figure 2.16: Phase space with respect to r

Chapter 3

Two-Dimensional Flows

3.1 Linear Systems

In one-dimensional systems, the dynamics are very limited—trajectories can only move in one direction or stay constant. In contrast, systems with two or more dimensions offer much richer behavior. To build up an understanding of this complexity, we start with the simplest case: linear systems in two dimensions.

These systems help us understand how fixed points behave in more complex nonlinear systems.

A two-dimensional linear system is of the form:

$$\dot{x} = ax + by,$$
$$\dot{y} = cx + dy,$$
$$\dot{X} = AX,$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Example 3.1 *Discuss the phase plane analysis of* $m\ddot{x} + kx = 0$.

Solution:

$$\dot{x} = v$$

$$\dot{v} = -(k/m)x$$

The system assigns a vector$(\dot{x}, \dot{v}) = (v, -\omega^2 x)$

As the system evolves, the phase point moves along a closed orbit. Suppose the motion begins at the farthest point on the left (large negative x, $v = 0$). The spring pulls the mass toward equilibrium, and v becomes positive. As x approaches zero, the velocity is maximal, and the mass overshoots the equilibrium position. It eventually reaches the opposite extreme ($x > 0$, $v = 0$), comes to rest momentarily, and is then pulled back again. This cycle continues indefinitely, tracing a closed curve in the phase plane.The phase portrait not only visualizes the motion but also encodes energy conservation geometrically (Figure: 3.2).

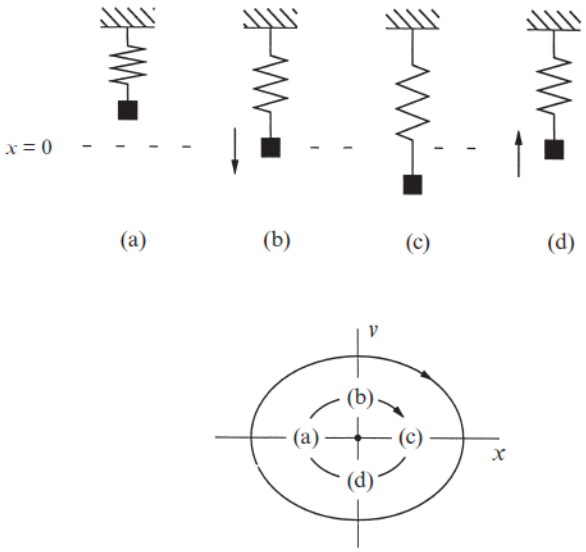


Figure 3.1: Visualization of simple harmonic motion with spring constant k

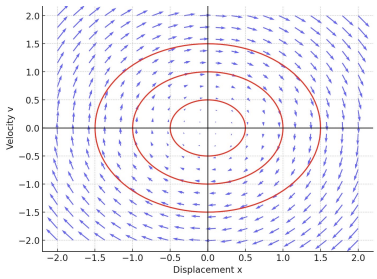


Figure 3.2: Phase Space of Simple harmonic motion

3.2 Classification of Linear Systems

We turn to the general case of a linear system governed by an arbitrary 2×2 matrix:

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Our goal is to classify all possible phase portraits for such systems.

3.2.1 Eigensolutions and Straight-Line Trajectories

In simpler examples, the coordinate axes often acted as invariant lines—trajectories starting on them stayed there. For a general matrix A, we look for analogous straight-line solutions of the form:

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}.$$

where λ is a scalar (growth rate) and $\mathbf{v} \neq 0$ is a fixed vector. Substituting into the differential equation yields:

$$\lambda e^{\lambda t}\mathbf{v} = Ae^{\lambda t}\mathbf{v} \Rightarrow A\mathbf{v} = \lambda\mathbf{v}$$

Thus, straight-line (exponential) trajectories exist when $\mathbf{v}$ is an eigenvector of A with eigenvalue λ. These are called **eigensolutions**.

3.2.2 Finding Eigenvalues and Eigenvectors

To find the eigenvalues, we solve the characteristic equation:

$$\det(A - \lambda I) = 0.$$

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, this becomes:

$$\lambda^2 - \tau\lambda + \Delta = 0$$

. where:

$$\tau = \text{trace}(A) = a + d, \quad \Delta = \det(A) = ad - bc$$

. Solving this quadratic gives the eigenvalues:

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

.

3.2.3 General Solution and Phase Portraits

If the eigenvalues $\lambda_1 \neq \lambda_2$ are real and distinct, their corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent and span the plane. Any initial condition $\mathbf{x}_0$ can then be expressed as:

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

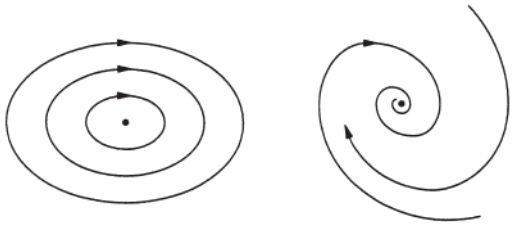
The general solution becomes:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

This is valid for all t, satisfies the differential equation.

3.2.4 Complex eigen values

- Eigenvalues are $\lambda_{1,2} = \frac{\tau}{2} \pm i\omega$, leads to oscillatory motion.
- Fixed points
  - Center (Neutral stability) , $\tau = 0$
  - Spiral: When $\tau < 0$, trajectories spiral inwards and vice versa.



(a) center (b) spiral

3.3 Classification of fixed points

- Saddle points: $\Delta > 0, \tau < 0$
- Non isolated fixed points: $\Delta = 0$
- Centers:$\Delta > 0, \tau = 0$
- Stable/Unstable Spirals: $\Delta < 0, \tau < 0 / \Delta < 0, \tau > 0$
- Stable/Unstable Nodes: $\Delta > 0, \tau > 0 / \Delta > 0, \tau < 0$

3.4 Vector Fields and Phase Portraits

A system of two differential equations can be written as:

$$\dot{x}_1 = f_1(x_1, x_2),$$
$$\dot{x}_2 = f_2(x_1, x_2),$$

or in vector form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

where $\mathbf{x} = (x_1, x_2)$ is the state of the system, and $\mathbf{f}(\mathbf{x})$ gives the direction of motion at that point.

Each starting point leads to a path called a *trajectory*. The collection of all these paths forms the *phase portrait*. For nonlinear systems, solving the equations exactly is usually hard, so we study their overall behavior instead.

3.4.1 Key Features of Phase Portraits

- **Fixed Points:** Points where $\dot{\mathbf{x}} = 0$. The system stays at rest here.
- **Closed Orbits:** Loops that repeat over time, showing periodic motion.
- **Nearby Behavior:** The way paths behave around fixed points and orbits tells us how the system evolves.

- **Stability:** Stable points attract nearby paths; unstable ones repel them.

Even without solving the system, these features help us understand how it behaves.

Summary

Ihis thesis is a review work of one-dimensional and two-dimensional dynamical system. First, we explored the behavior of dynamical systems using both analytical and qualitative approaches. We began with the meaning and evolution of dynamics. Further, we explored one-dimensional flows, focusing on fixed points, stability, bifurcations, and real-world examples like population models and lasers. Through phase line and potential analysis, we saw how even simple equations can show such a varied behavior depending on system parameters. The variation of parameter plays so vital role in the evolution of system over time.

We then extended our study to two-dimensional systems, starting with linear systems. Using eigenvalues and eigenvectors, we classified different types of fixed points and sketched phase portraits to understand the global behavior of solutions. We also studied vector fields and learned how to analyze nonlinear systems without solving them explicitly.

The key takeaway is that while nonlinear systems are often difficult to solve exactly, tools like phase plane analysis and bifurcation theory allow us to understand their long-term and qualitative behavior of the system. These methods are widely applicable in fields such as physics, biology, and engineering, wherever systems evolve over time.

Overall, this study provides a strong foundation for further work in dynamical systems, including higher-dimensional and chaotic systems.

Future Works

This dissertation provides a foundational study of dynamical systems using one- and two-dimensional models to understand fixed points, stability, bifurcations, and phase portraits. However, dynamical systems theory offers a vast field of exploration beyond these initial dimensions. Future work may consider the following directions:

- To explore more on two-dimensional system including physical examples.
- Higher-Dimensional Systems: Extending the analysis to systems with three or more variables can reveal richer dynamics, including chaos and strange attractors, which are not present in lower-dimensional flows. This would include a study of Lorenz systems, Rössler systems, and models from fluid dynamics or weather systems.
- Limit Cycles and Hopf Bifurcation: A deeper study into periodic orbits in two-dimensional nonlinear systems, including the application of the Poincaré–Bendixson Theorem, can provide further insight into biological and chemical oscillators.

Bibliography

[1] Steven H. Strogatz."Non Linear Dynamics And Chaos",With Applications to Physics,Biology,Chemistry and Engineering.Second Edition,2018.